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CONTRACTION POLYNOMIAL

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ABSTRACT

Polynomial is one of the most natural association with graphs. A number of polynomials such as reliability polynomial, Tutte polynomial, characteristic polynomial, chromatic polynomial for some graphs have been determined.In this paper, the contraction polynomial of some families of graphs are determined.

Keywords: contraction polynomial, partition, shrink.

INTRODUCTION

Algebraic graph theory can be viewed as an extension to graph theory in which algebraic methods are applied to problems about graphs. The expanding area of algebraic graph theory uses different branches of algebra to explore various aspects of graph theory. Matrices and polynomials have a highly natural association with graphs. A number of polynomials such as reliability polynomial, Tutte polynomial, characteristic polynomial, chromatic polynomial for some graphs have been determined. In this paper, we deals with contraction Polynomial for some families of graphs.

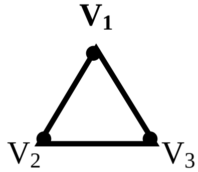
DEFINITION: [Acharya, 2012]

Let G=(V,E) be a graph and π= {$V_1, V_2, V_3, \ldots V_k$} any partition of V(G). The graph π (G) whose vertex set is π and whose edges are defined by the rule $V_iV_j \in E(\pi(G))$ iff $\exists$ u$\in V_i$ and v$\in V_j$ with uv$\in$ E(G) is called shrink of G with respect to π. A contraction of G is a partition π of V(G) in which the subgraph induced by V_i , denoted $< V_i >$ is connected for each i $\in$ **k**={1,2,..k} here, k is called the order of π.For convenience, by a 'contraction' of G we shall mean either a partition π of V(G) which is a contraction of G or the shrink π (G) of G with respect to π. Let ψ_G denote the set of contractions of G and let $\psi_{G,r}$ denote the set of contractions of order r in ψ_G. If p(x) and q(x) denote respectively the order and size of the graph X, call the polynomial

$c(G;x,y)=\sum_{X \epsilon \psi_G} x^{p(X)} y^{q(X)}$ in two formal variables x and y, the contraction polynomial of G.

Example:

Consider the graph K_3



The possible partitions are

(i) π_1={{v_1}, {v_2},{v_3}}

(ii) π_2= {{v_1}, {v_2, v_3}}

(iii) π_3= {{v_2}, {v_1, v_3}}

(iv) π_4= {{v_3}, {v_1, v_2}}

(v) π_5= {v_1, v_2, v_3}, then the contraction of K_3 is K_1

$\pi_1(K_3)=K_3$; $\pi_2(K_3)=K_2$;

$\pi_3(K_3)=K_2$; $\pi_4(K_3)=K_2$; $\pi_5(K_3)=K_1$;

ψ_G = { $\pi_1, \pi_2, \pi_3, \pi_4, \pi_5$}

Hence c (K_3;x,y)= $x^3y^3 +3 x^2y+x$

Stirling number of second kind:[Jozsef sandor and Borislav cristici, 2004]

Stirling number of second kind is the number of ways to partition a set of p objects into r non empty subsets and it is denoted by S (p,r)

The stirling number, S (p,r) of second kind is given by,

$S(p,r)=\frac{1}{r!}\sum_{i=0}^{r}(-1)^i\binom{r}{i}(r-i)^n.$

Stirling numbers of second kind of certain values are tabulated below

p/r	0	1	2	3	4	5
0	1					
1	0	1				
2	0	1	1			
3	0	1	3	1		
4	0	1	7	6	1	
5	0	1	15	25	10	1
6	0	1	31	90	65	15
7	0	1	63	301	350	140

Theorem 1:

For any positive integer p,

$c(K_p;x,y)=\sum_{r=1}^{p} S(p,r)x^r y^{\binom{r}{2}}$, where S(p,r) denotes the stirling number of second kind.

Proof:

If π is a partition of $V(K_p)$ with k parts, then the shrink π (K_p) will be a complete graph with k vertices.

Number of partitions with k parts is equal to S (p,k)

Hence, $c(K_p;x,y)=\sum_{r=1}^{p} S(p,r)x^r y^{\binom{r}{2}}$

Theorem: 2

For any positive integer p,

$c(\overline{K_p};x,y)=\sum_{r=1}^{p} S(p,r)x^r$, where S(p,r) denotes the stirling number of the second kind.

Proof:

If π is a partition of V $(\overline{K_p})$ with k parts, then the shrink π $(\overline{K_p})$ will be an empty graph with k vertices.

Number of partitions with k parts is equal to S (p,k)

Hence, $c(\overline{K_p};x,y)=\sum_{r=1}^{p} S(p,r)x^r$

Theorem 3:

For any positive integer p,

$c(K_{1,p};x,y)=\sum_{r=1}^{p} S(p,r)x^r y^{(r-1)}$, where S(p,r) denotes the stirling number of the second kind.

Proof:

If π is a partition of V $(K_{1,p})$ with k parts, then the shrink π $(K_{1,p})$ will be a star graph with k vertices.

Number of partitions with k parts is equal to S (p,k)

Hence, $c(K_{1,p};x,y)=\sum_{r=1}^{p} S(p,r)x^r y^{(r-1)}$

Illustration:

Consider the graph K_6,

Let $V(K_6) = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

Number of partition of 6 is 11

The partition of 6 possibilities of $V(K_6)$

(i) $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\},\{v_6\}\}$

(ii)$\{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}\}$

(iii)$\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5, v_6\}\}$

$\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}, \{v_6\}\}$

(iv) $\{\{v_1\}, \{v_2\}, \{v_3, v_4, v_5, v_6\}\}$

$\{\{v_1, v_2\}, \{v_3, v_4, v_5\}, \{v_6\}\}$

$\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5, v_6\}\}$

(v) $\{\{v_1\}, \{v_2, v_3, v_4, v_5, v_6\}\}$

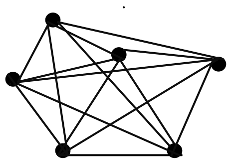
$\{\{v_1, v_2\}, \{v_3, v_4, v_5, v_6\}\}$

$\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$

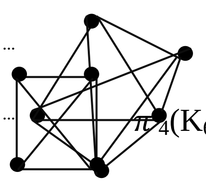
$\{\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\}$

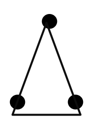
(vi) $\{v_1, v_2, v_3, v_4, v_5, v_6,\}$

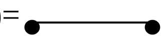
Then the contractions of K_6 are

$\pi_1(K_6)=$ 

$\pi_2(K_6)=$

$\pi_3(K_6)=$ 

$\pi_4(K_6)=$ 

$\pi_5(K_6)=$  $\pi_6(K_6)=$ 

Then $c(K_6; x, y) = \sum_{r=1}^{6} S(6,r)x^r y^{\binom{r}{2}}$

$= S(6,2)x^2y + S(6,3)x^3y^3 + S(6,4)x^4y^6$

$+ S(6,5)x^5y^{10} + S(6,6)x^6y^{15}$

$=31x^2y + 90x^3y^3 + 65x^4y^6$

$+15x^5y^{10} + x^6y^{15} + x$

Theorem: 4

In the contraction polynomial of complete bipartite graph $K_{m,n}$, the highest power for x is m+n and that for y is mn.

Proof:

In complete bipartite graph $K_{m,n}$, number of vertices is m+n and number of edges is mn.

The number of edges in any contraction should be less than or equal to number of edges in $K_{m,n}$. Therefore power of y in contraction polynomial of $K_{m,n}$ is less than or equal to mn.

For the partition {1, 1, 1,…… (m+n) times1} the corresponding contraction has mn edges.

Therefore highest power of y in the contraction polynomial of complete bipartite graph $K_{m,n}$ is mn.

Number of vertices is maximum for this partition (1, 1, 1,… (m+n) times 1) only.

Therefore highest power of x in the contraction polynomial of complete bipartite graph $K_{m,n}$ is m+n.

Theorem: 5

In the contraction polynomial of complete bipartite graph $K_{m,n}$ the coefficient of x^2y is S(m+n,2).

Proof:

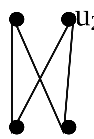
The contraction of $K_{m,n}$ with 2 parts is K_2.

Since contraction of $K_{m,n}$ contains 2 vertices and one edge.

The number of partition with 2 parts is equal to S(m+n,2).Hence the coefficient of x^2y is equal to S(m+n,2).

Illustration:

Consider the graph $K_{2,2}$,



The partition of $K_{2,2}$ is

(i) $\{\{u_1\}, \{u_2\}, \{v_1\}, \{v_2\}\}$

(ii) $\{\{u_1\}, \{u_2, v_1, v_2\}\}$

$\{\{v_1\}, \{u_1, u_2, v_2\}\}$

$\{\{u_1, u_2\}, \{v_1, v_2\}\}$

$\{\{u_1, v_1\}, \{u_2, v_2\}\}$

(iii) $\{\{u_1\}\{u_2\}, \{v_1, v_2\}\}$

$\{\{u_1, u_2\}, \{v_1\}, \{v_2\}\}$

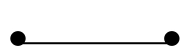
$\{\{v_1, u_1\}, \{u_2\}, \{v_2\}\}$

$\{\{u_1\}\{v_1\}, \{u_2, v_2\}\}$

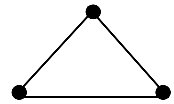
(iv) $\{u_1, v_1, u_2, v_2\}$

In (i) the contraction of the graph is $K_{2,2}$

In (ii) the contraction of the graph is K_2

$\pi_2(K_{2,2})=$ 

In (iii) the contraction of the graph is K_3

$\pi_3(K_{2,2})=$ 

In (iv) the contraction of the graph is K_1.

$\pi_4(K_{2,2})$ = 

Then c $(K_{2,2}; x, y)$ = $x+7x^2y+2x^3y^2+4x^3y^3+x^4y^4$

The contraction polynomial for $K_{2,n}$ ($n\leq$ 5)has been tabulated below.

$K_{2,2}$	$x+7x^2y+2x^3y^2+4x^3y^3+x^4y^4$
$K_{2,3}$	$x+15x^2y+22x^3y^2+3x^3y^3+$ $x^4y^3+3x^4y^4+6x^4y^5+x^5y^6$
$K_{2,4}$	$x+31x^2y+84x^3y^2+6x^3y^3+$ $21x^4y^3+22x^4y^4+12x^4y^5+x^5y^4$ $+6x^5y^6+8x^5y^7+x^6y^8$
$K_{2,5}$	$x+63x^2y+90x^3y^2+211x^3y^3+$ $+265x^4y^4+65x^4y^3+20x^4y^5$ $+15x^5y^4+25x^5y^6+100x^5y^7$ $x^6y^5+10x^6y^8+10x^6y^9$

Theorem: 6

In the contraction polynomial of $P_n.K_1$, path of length n the highest power for x is n+2 and that for y is 2n+1.

Proof: In the graph $P_n.K_1$,

Number of vertices is n+2

Number of edges is 2n+1.

The number of edges in any contraction should be less than or equal to number of edges in $P_n.K_1$.

Therefore power of y in contraction polynomial of $P_n.K_1$ is less than or equal to 2n+1.For the partition {1,1,1,……(n+2) times1}the corresponding contraction has 2n+1 edges.

Therefore highest power y in the contraction polynomial of $P_n.K_1$ is 2n+1.

Number of vertices is maximum for this partition {1,1,1,….(n+2)times1} only.

Therefore highest power of x in the contraction polynomial of $P_n.K_1$ is n+2.

Theorem: 7

In the contraction polynomial of $P_n.K_1$, the coefficient of x^2y is S (n+2, 2).

Proof: The contraction of $P_n.K_1$ with 2 parts is K_2 and the number of partition with 2 parts is equal to S(n+2, 2).

Hence the coefficient of x^2y is S (n+2, 2).

The contraction polynomial for $P_n.K_1$ (n≤ 5) has been tabulated below.

$c(P_1.K_1;x,y)$	$x+3x^2y+x^3y^3$
$c(P_2.K_1;x,y)$	$x+7x^2y+x^3y^2+5x^3y^3+x^4y^5$
$c(P_3.K_1;x,y)$	$x+15x^2y+5x^3y^2$ $20x^3y^3+2x^4y^4+8x^4y^5+x^5y^7$
$c(P_4.K_1;x,y)$	$x+31x^2y+18x^3y^2 72x^3y^3+x^4y^3+15x^4y^4+45x^4y^5+4x^4y^6+3x^5y^6+12x^5y^7+x^6y^9$
$c(P_5.K_1;x,y)$	$x+63x^2y+56x^3y^2+245x^3y^3+72x^4y^4+x^4y^5+61x^4y^6+x^5y^4+25x^5y^6+79x^5y^7+32x^5y^8+3x^5y^9+4x^6y^8+13x^6y^9+4x^6y^{10}+x^7y^{11}$

The contraction polynomial for $S_n.K_1$ (n≤ 5) has been tabulated below.

$c(S_1.K_1;x,y)$	$x+3x^2y+x^3y^3$
$c(S_2.K_1;x,y)$	$x+7x^2y+x^3y^2+5x^3y^3+x^4y^5$
$c(S_3.K_1;x,y)$	$x+15x^2y+16x^3y^2 19x^3y^3+x^4y^3+9x^4y^5+x^5y^7$
$c(S_4.K_1;x,y)$	$x+31x^2y+25x^3y^2+10x^4y^3+55x^4y^5+x^5y^4+14x^5y^7+x^6y^9$
$c(S_5.K_1;x,y)$	$x+63x^2y+89x^3y^2+212x^3y^3+x^4y^2+66x^4y^3+283x^4y^5+16x^5y^4+124x^5y^7+x^6y^5+20x^6y^9+x^7y^{11}$

CONCLUSION

Getting recursive relation for contraction polynomial is our main target. It is strongly hoped that it will be attained atleast for some families of graphs.

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