ROLE OF $\alpha_1$ AND $\alpha_2$ NEAR RING IN BOOLEAN $S$-NEAR RING

$^1$Radha.D and $^2$Dhivya.C

$^1$,$^2$Assistant Professor, PG and Research Department of Mathematics,

$^1$,$^2$A.P.C.Mahalaxmi College for Women, Thoothukudi, Tamilnadu, India.

Corresponding Author: radharavimaths@gmail.com

ABSTRACT:

In this paper we have proved some results on Boolean $S$-near ring using the concepts of regular near ring, idempotents, left cancellation law etc. It is proved that $N$ is a $S$-near ring iff $N$ is boolean whenever $N$ is regular. Every Boolean $S$-near ring is both $\alpha_1$ and $\alpha_2$ near ring with the converse in the case of $\alpha_2$ near ring. Also, as a characterization theorem it is proved that a Boolean regular near ring is an $S$-near ring in each of the following cases (i) $N$ is an IFP with identity (ii) $Na = aNa$ for all $a \in N$ (iii) $N$ is subcommutative.

Mathematics Subject Classification: 16Y30

Keywords: $\alpha_1$ near ring, $\alpha_2$ near ring, strongly regular, subcommutativity, $S_1$ near ring, $S_2$ near ring.

INTRODUCTION

Near rings can be thought of as generalized rings: if in a ring we ignore the commutativity of addition and one distributive law, we get a near ring. Gunter Pilz [2] "Near rings" is an extensive collection of the work done in the area of near rings.

A near ring $N$ is a system $(N, +, \cdot)$ such that $(N, +)$ is a group (not necessarily abelian), $(N, \cdot)$ is a semigroup, the right distributive law holds, i.e. $(x + y)z = xz +
yz for each x, y, z in N; and x \cdot 0 = 0 for every x in N [6]. A near ring N is an S-near ring if \( a \in Na \) for each \( a \in N \) [6]. Let N be a right near ring, if (i) for every a in N there exists x in N such that \( a = xax \) then we say N is an \( \alpha_1 \) near ring. (ii) for every a in \( N^* \) there exists x in \( N^* \) such that \( x = xax \) then we say N is \( \alpha_2 \) near ring [27].

**Preliminaries**

**Definition 2.1** [4]

The near rings N are **boolean** if \( x^2 = x \) for each \( x \in N \).

**Definition 2.2** [6]

A near ring N is defined to be **left bipotent** if \( Na = Na^2 \) for each \( a \in N \).

**Definition 2.3** [6] A near ring N is **regular** if for each \( a \) in N, there exists \( x \) in N such that \( a = axa \).

**Definition 2.4** [3]

If all non zero elements of N are left(right) cancellable then we say that N fulfills the left(right) cancellation law.

**Notation 2.5** [25]

\( E \) denotes the set of all idempotent of N (\( a \in E \) iff \( a^2 = a \)).

**Definition 2.6**[3]

N is said to fulfill the **Insertion of Factors Property (IFP)** provided that for all \( a, b, n \) in N, \( ab = 0 \Rightarrow anb = 0 \).

**Definition 2.7** [1]

N is called a \( P_k \) near ring (\( P_k' \) near ring) if there exists a positive integer \( k \) such that \( x^kN = xNx \) \( (Nx^k = xNx) \) for all \( x \in N \).

**Definition 2.8** [8]

N is said to be **subcommutative** if \( Na = aN \) for all \( a \in N \).

**Notation 2.9** [25]

\( N^* \) denotes the set of all nonzero elements of N, i.e., \( N^* = N - \{0\} \).

**Definition 2.10** [25]

N is called an \( S_1 \) near ring (\( S_2 \) near ring) if for every \( a \in N \), there exists \( x \in N^* \) such that \( axa = xa \) \( (axa = ax) \).

**Lemma 2.11** [4]

If N is a boolean near ring, then \( xy = xyx \) for each \( x, y \in N \).

**Definition 2.12** [2]

A near ring N is said to be **strongly regular** if for each \( a \in N \), there exists an element \( x \in N \) such that \( a = xa^2 \).

**Main Results**

**Theorem 3.1**

Let N be a reduced near ring. N is left bipotent iff N is boolean.

**Proof:**

Let N be left bipotent. Then \( Na = Na^2 \) for each \( a \) in N. \( \Rightarrow xa = xa^2 \) for all \( x \) in N. \( \Rightarrow xa - xa^2 = 0 \). \( \Rightarrow x(a - a^2) = 0 \). \( \Rightarrow a - a^2 = 0 \), since N is reduced. This
gives $a = a^2$. Hence $N$ is boolean. Converse follows.

**Theorem 3.2**

Let $N$ be a regular near ring. $N$ is $S$-near ring iff $N$ is boolean.

**Proof:**

Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x \in N$. Since $N$ is regular, for each $a \in N$, there exists $x \in N$ such that $a = axa$. This gives $a = a \cdot a = a^2$. Therefore $a = a^2$. Hence $N$ is boolean. Conversely, let $N$ be regular. Then for each $a \in N$, there exists $x \in N$ such that $a = xa$. Since $N$ is boolean, $a^2 = a$. This gives $a^2 = axa$. By left cancellation law, $a = xa$. Therefore $a \in Na$. Hence $N$ is $S$-near ring.

**Theorem 3.3**

Let $N$ be boolean near ring. If $N$ is $S$-near ring then $N$ is regular.

**Proof:**

Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x$ in $N$. Since $N$ is boolean, $a^2 = a \cdot a = axa$. Therefore $a = axa$. Hence $N$ is regular.

**Theorem 3.4**

Let $N$ be $S$-near ring. If $xa = 0$ then $ax = 0$ for all $a \in N$ and for some $x \in N$.

**Proof:**

Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x \in N$. Now $ax = xax = 0x = 0$. Hence $ax = 0$.

**Theorem 3.5**

Let $N$ be $S$-near ring. If $N$ is boolean, then

(i) $ax \in E$

(ii) If the left cancellation law is valid in $N$ then $xa \in E$ for all $a \in N$ and for some $x \in N$.

**Proof:**

Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x$ in $N$. Let $N$ be boolean. Then $a^2 = a$ for all $a \in N$. (i) $(ax)^2 = (ax)(ax) = aax = a^2x$ (Since $N$ is boolean). That is $(ax)^2 = ax$ and hence $ax \in E$. (ii) Consider $a(xa)^2 = a(xa)(xa) = a(a(xa)) = a^2(xa) = axa$ (Since $N$ is boolean). Therefore $a(xa)^2 = axa$. Since the left cancellation is valid in $N$, $(xa)^2 = xa$. Thus $xa \in E$.

**Theorem 3.6**

Let $N$ be subcommutative and $S$-near ring. If $N$ is boolean then $N$ is strongly regular.

**Proof:**

Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x$ in $N$. Since $N$ is subcommutative, $Na = aN$. Therefore for any $x \in N$, there exists $y \in N$ such that $xa = ay$. This implies $a = ay$. Now $ay = xa \Rightarrow aya = xaa = xa^2 \Rightarrow$
 ROLE OF $\alpha_1$ AND $\alpha_2$ NEAR RING IN BOOLEAN S-NEAR RING

$aa = xa^2 \Rightarrow a^2 = xa^2 \Rightarrow a = xa^2$ (Since $N$ is boolean). Hence $N$ is strongly regular.

**Theorem 3.7**
Let $N$ be $S$-near ring. If $N$ is strongly regular then $N$ is boolean.

**Proof:**
Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x \in N$. Since $N$ is strongly regular, for each $a \in N$, there exists an element $x \in N$ such that $a = xa^2$. This implies $a = xaa = a^2$. Therefore $a = a^2$. Hence $N$ is boolean.

**Theorem 3.8**
Let $N$ be $S$-near ring. If $N$ is boolean, then $N$ is $\alpha_1$ near ring.

**Proof:**
Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x$ in $N$. Since $N$ is boolean, by lemma 2.1 we have $xa = xax$ for each $x, a \in N$. This implies $a = xax$. Hence $N$ is $\alpha_1$ near ring.

**Theorem 3.9**
Let $N$ be $S$-near ring. $N$ is boolean iff $N$ is $\alpha_2$ near ring.

**Proof:**
Let $N$ be $S$-near ring. Then $x \in Nx$ for all $x \in N$. This implies $x = ax$ for some $a$ in $N$. Since $N$ is boolean, $x = x^2 = xax$. Therefore $x = xax$. In particular $x = xax$ for any $x, a \in N^*$. Hence $N$ is $\alpha_2$ near ring. Conversely, since $N$ is $\alpha_2$ near ring, for every $a$ in $N^*$ there exists $x$ in $N^*$ such that $x = xax$. This implies $x = xx = x^2$. Therefore $x = x^2$. Hence $N$ is boolean.

**Theorem 3.10**
Let $N$ be boolean near ring. If $N$ is commutative then $N$ is $S_1$ near ring.

**Proof:**
Since $N$ is boolean, by lemma 2.1 we have $ax = axa$ for each $a, x \in N$ which gives $xa = axa$, since $N$ is commutative. In particular, $xa = axa$ for any $x \in N^*$. Hence $N$ is $S_1$ near ring.

**Theorem 3.11**
Let $N$ be $S$-near ring. $N$ is regular iff $N$ is a $S_1$ near ring.

**Proof:**
Let $N$ be $S$-near ring. Then $a \in Na$ for all $a \in N$. This implies $a = xa$ for some $x$ in $N$. Since $N$ is regular, for each $a$ in $N$, there exists $x$ in $N$ such that $a = axa$ which gives $axa = xa$. In particular $axa = xa$ for any $x \in N^*$. Hence $N$ is $S_1$ near ring. Conversely, since $N$ is $S_1$ near ring, for every $a \in N$, there exists $x \in N^*$ such that $axa = xa$ which gives $axa = a$. Hence $N$ is regular.

**Corollary 3.12**
If $N$ is boolean, then $N$ is $S_2$ near ring.

**Theorem 3.13**
Let $N$ be a boolean near ring. If $N$ is regular, then each of the following statements
implies that $N$ is an $S$-near ring. (i) $N$ is an IFP near ring with identity. (ii) $Na = aNa$ for all $a \in N$. (iii) $N$ is subcommutative. (iv) $N$ is zero symmetric.

**Proof:**

Since $N$ is regular, for each $a \in N$, there exists $x \in N$ such that $a = axa$. (i) Let $N$ be an IFP near ring with identity '1' and let $a \in N$. Since $N$ is boolean, $a^2 = a \Rightarrow a^2 - a = 0 \Rightarrow (a - 1)a = 0$. Since $N$ has IFP, $(a - 1)xa = 0$ for all $x \in N$. \(\Rightarrow axa - xa = 0 \Rightarrow a - xa = 0 \Rightarrow a = xa \Rightarrow a \in Na\) for all $x \in N$. Hence $N$ is an $S$-near ring.

(ii) Since $Na = aNa$, for any $x \in N$, there exists $y \in N$ such that $xa = aya$. Now $axa = a(xa) = a(aya)$

\[= a^2ya = aya = xa.\]

Hence $axa = xa$. This implies $a = xa$. Therefore $a \in Na$. Hence $N$ is an $S$-near ring. (iii) Since $N$ is subcommutative, $Na = aN$. Therefore for any $x \in N$, there exists $y \in N$ such that $xa = ay$. Therefore $axa = a(xa) = a(ay) = a^2y = ay$. That is $axa = ay$. This implies $a = ay$ which gives $a = xa$. Therefore $a \in Na$. Hence $N$ is an $S$-near ring. (iv) Let $N$ be zero symmetric near ring. Let $a \in N$. If $a \neq 0$, we take $x = a$. Then $axa = a^2a = xa$. This gives $a = xa$. If $a = 0$ then for any $x \in N$, $a = 0 = xa$. Hence $N$ is $S$-near ring.

**REFERENCE**

ROLE OF $\alpha_1$ AND $\alpha_2$ NEAR RING IN BOOLEAN S-NEAR RING


